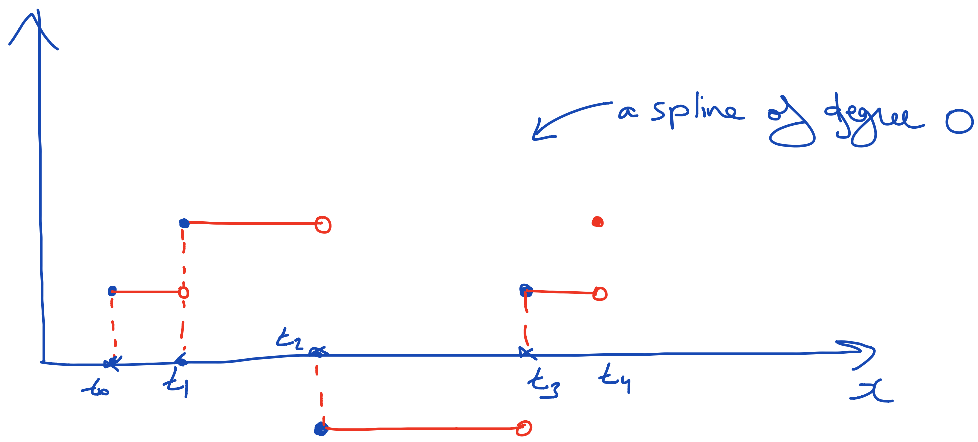
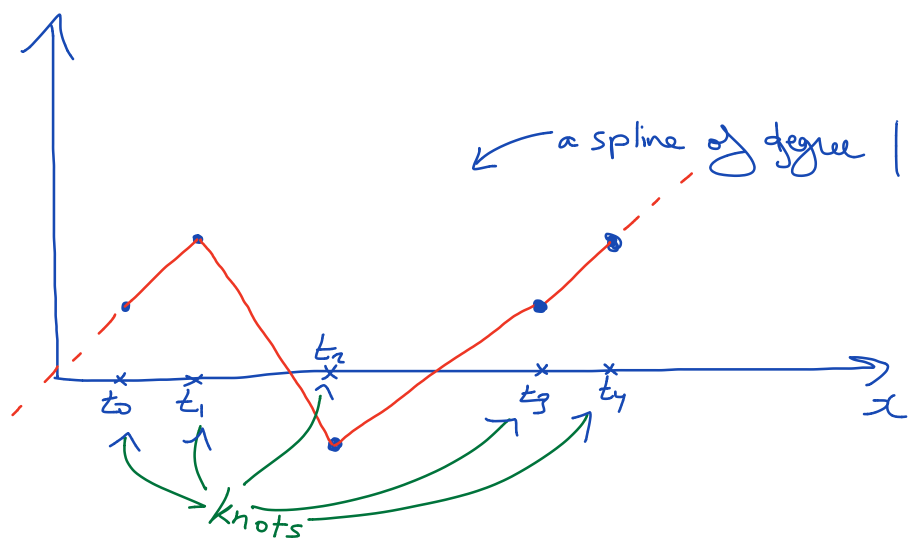
Spline Interpolation





A spline function consists of polynomial pieces on subintervals. They are joined together in a way that satisfies continuity conditions.

- on $[t_{i-1}, t_i)$, S is a polynomial of degree $\leq k$
- S has a continuous $(k-1)^{st}$ derivative on $[t_0, t_n]$

e.g. Spline of degree 0 : piecewise constant
Spline of degree 1 : piecewise linear

Cubic Splines (Splines with degree $k=3$)

Denote by $S(x)$ the cubic spline interpolating some data (t_i, y_i), $0 \leq i \leq n$

$$S(x) = \begin{cases} S_0(x) & x \in [t_0, t_1] \\ S_1(x) & x \in [t_1, t_2] \\ \vdots & \vdots \\ S_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}$$

so $S_{i-1}(t_i) = S_i(t_i) \; (= y_i)$ for $i = 1, \dots, n-1$

we also ask that S' & S'' are continuous

Is this possible?

- we have 2-equations per S_i : $S_i(t_i) = y_i$ & $S_i(t_{i+1}) = y_{i+1}$

$\Rightarrow 2n$ equations

- we also have $S_i'(t_i) = S_{i-1}'(t_i)$ for $i = 1, -, n-1$

$\Rightarrow 2(n-1)$ equations $S_i''(t_i) = S_{i-1}''(t_i)$ for $i = 1, -, n-1$

$\Rightarrow$ Total of $4n-2$ equations

But each $S_i(x)$ has 4-degrees of freedom

$(a+bx+cx^2+dx^3)$

$\Rightarrow$ $4n$-unknowns

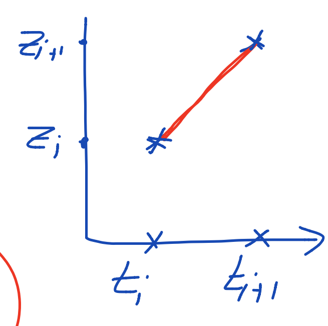
We're left with 2 degrees of freedom.

OK, let's now figure out the S_i's.

- Let $z_i = S_i''(t_i)$ $\quad \left(z_i = \lim_{x \to t_i^+} S''(x) = \lim_{x \to t_i^-} S''(x) \quad i = 1, \dots, n-1 \right)$

moreover $S_i''(x)$ is a linear function with $S_i''(t_i) = z_i$, $S_i''(t_{i+1}) = z_{i+1}$

So



$$S_i''(x) = \frac{t_{i+1} - x}{\underbrace{t_{i+1} - t_i}_{h_i}} \cdot z_i + \frac{x - t_i}{t_{i+1} - t_i} \cdot z_{i+1}$$

$$\Rightarrow \boxed{S_i(x) = \frac{z_i}{6h_i}(t_{i+1}-x)^3 + \frac{z_{i+1}}{6h_i}(x-t_i)^3 + C(x-t_i) + D(t_i - x)}$$

integrate twice

constants of integration

To find C & D: $S_i(t_i) = y_i$, $S_i(t_{i+1}) = y_{i+1}$

$$\Rightarrow \boxed{C = \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}\right) \ \& \ D = \frac{y_i}{h_i} - \frac{z_i h_i}{6}}$$

- Now, to find the z_i's note that $S'_{i-1}(t_i) = S'_i(t_i)$

Differentiate the expressions for $S_i(x)$, $S_{i-1}(x)$ and plug in t_i :

$$\Rightarrow h_{i-1}z_{i-1} + 2(h_i + h_{i-1})z_i + h_i z_{i+1} = \frac{6}{h_i}(y_{i+1} - y_i) - \frac{6}{h_{i-1}}(y_i - y_{i-1})$$

for $i = 1, \ldots, n-1$

$\Rightarrow$ $n+1$ unknowns $(z_0 \text{---} z_n)$
$n-1$ equations

Simply set $z_0 = 0$, $z_n = 0$ and solve for the rest
$\Rightarrow$ natural cubic spline

Theorem: Let $f \in C^2[a,b]$ & let

$a = t_0 < t_1 < t_2 \leq \cdots t_n = b$. If S is the natural cubic spline interpolating f at t_i, $i = 0, \to n-1$

then $$\int_a^b (S''(x))^2 dx \leq \int_a^b f''(x)^2 dx$$

↖ among all f's that fit the data S is the "smoothest"

Proof

Let $g := f - s \iff f = g + s$

$$\Rightarrow \int_a^b (f''(x))^2 dx = \underbrace{\int_a^b (g''(x))^2 dx}_{\geq 0} + \int_a^b (s''(x))^2 dx + 2\underbrace{\int_a^b g''(x) s''(x) dx}_{? \geq 0 ?}$$

If we can show that $\int_a^b g'' s'' dx$ is ≥ 0 we are done

$$\int_a^b g''(x) s''(x) dx = \sum_{i=0}^{n} \int_{t_{i-1}}^{t_i} g''(x) s''(x) dx$$

Integration by parts: $(g' s'')' = g'' s'' + g' s'''$

$$\Rightarrow \int_{t_{i-1}}^{t_i} g'' s'' = \int_{t_{i-1}}^{t_i} (g' s'')' - \int_{t_{i-1}}^{t_i} g' s''' = (g's'')(t_i) - (g's'')(t_{i-1})$$

$$- \int_{t_{i-1}}^{t_i} g' s'''$$

$$\Rightarrow \sum_{i=0}^{n} \int_{t_i}^{t_i} g''(x)\, s''(x)\, dx = \overbrace{\sum_{i=0}^{n} (g's'')(t_i) - (g's'')(t_{i-1})}^{(g's'')(b) - (g's'')(a) = 0}$$

bec $s''(a) = s''(b) = 0$

$$- \sum_{i=0}^{n} \int_{t_{i-1}}^{t_i} g' s_i'''$$

$$= - \sum_{i=0}^{n} c_i \int_{t_{i-1}}^{t_i} g'(x)\, dx = - \sum_{i=0}^{n} c_i \Big[\underbrace{g(t_i)}_{\substack{f(t_i) - s(t_i) \\ = 0}} - \underbrace{g(t_{i-1})}_{=0} \Big]$$

$= 0$!